The Equations used in Comparing Electric Air Taxi Visions

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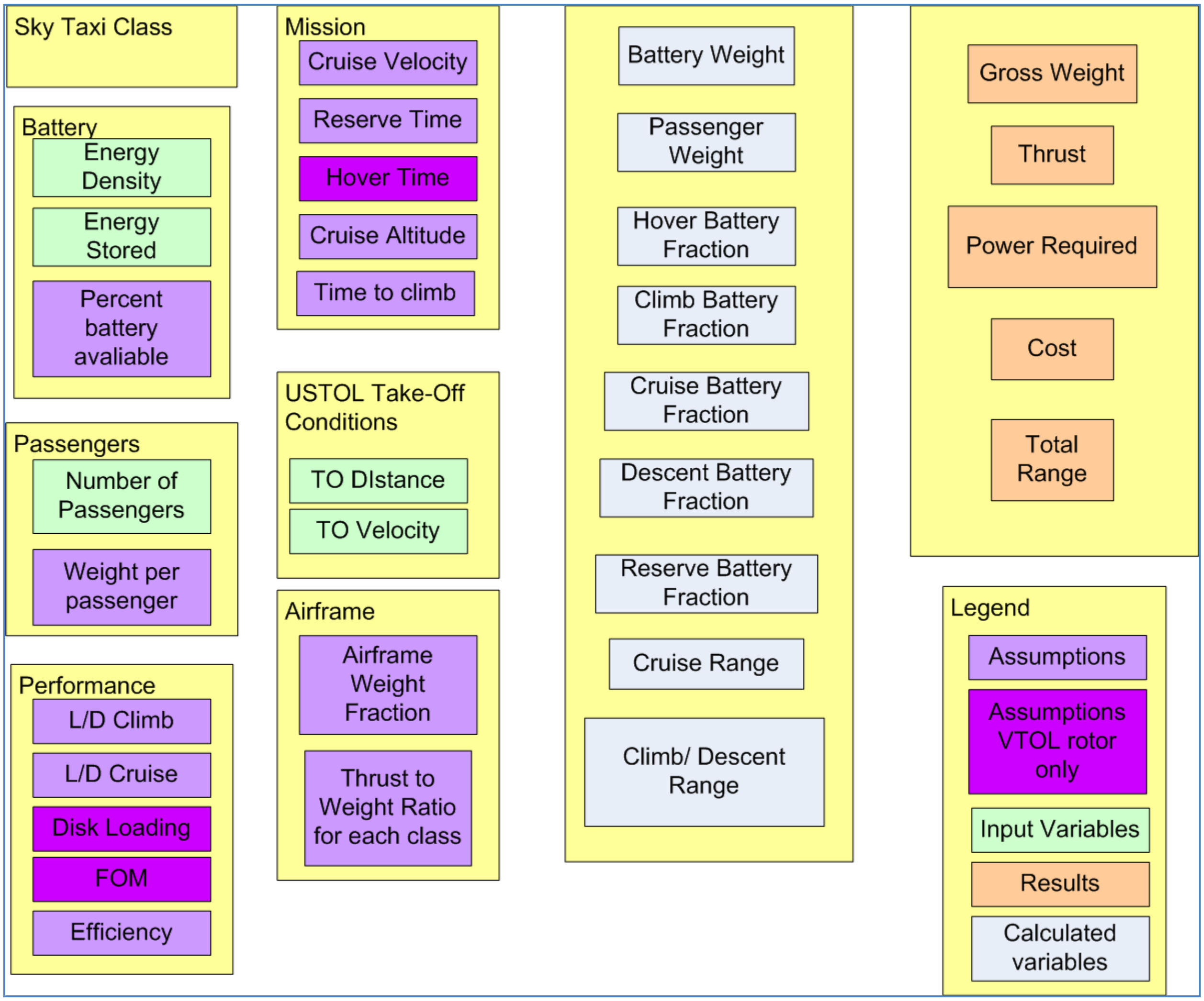
The models used in “Comparing Electric Air Taxi Visions” are developed here.

Variables

ρ_a = air density (taken as sea level)
ρ_b = battery energy density of 150, 300, 450 and 600 wh/kg
η = total combined efficiency from batteries to useful thrust (.85)
AGL = cruise altitude
CL_{to} = Take-off lift coefficient
ROC = Rate of climb
DL = Disk loading (lb/ft^2)
Ec = Battery capacity (kwh)
E_h = however energy
L/D_{climb} = climb L/D ratio
L/D_{cruise} = cruise lift to drag ratio
Np = number of passengers
P_h = Power for hover (ft lb/sec)
R = Range is in miles
Sg = ground roll for take-off
tc = time to climb (sec)
tcr = cruise time (sec)
t_h = hover time (sec)
tr = reserve time (sec)
Vc = V_{cruise} = Cruise velocity (ft/sec)
Vto = take-off velocity
Wo = total weight (lb)
Wa = airframe weight including all electronics and motors, excluding batteries (lb)
Wbt = total battery weight (lb)
Wbto = battery weight for taxi and takeoff (lb)
Wbh = battery weight for hover (lb)
Wbc = battery weight for climb (lb)
Wbcr = battery weight for cruise (lb)
Wbd = battery weight for descent (lb)
Wblt = battery weight for landing and taxi (lb)
Wbr = battery weight reserve (lb)
Wp = payload weight (assume 220 lb (100kg) per passenger.



The Input And Calculated Variables.

Basic logic - for each combination of:

- Sky taxi class
- Battery energy density
- Battery capacity (energy stored)
- Number of passengers

And these assumptions (rational ad sensitivity given in paper):

- Cruise velocity (mph/fps/kts) Vc=150/220/130
- Cruise altitude (ft/m) AGL = 1000/300
- Climb rate (CR) 500 ft/min (2.5 m/sec) then time to climb (tc) is 2 minutes
- Electrical and propulsor efficiency = 0.76
- Airframe weight fraction including motors and controllers = 50%
- Passenger and luggage weight (kg/lb) = 100/220

- Factor of Merit (FOM) = 0.7
- Hover time (sec) = 60 (120 sec explored in sensitivity analysis)
- Reserve time (min) = 10
- Percent battery capacity available= 80%
- Battery cost ($/kwh): Current = $200, Near future = $100
- USTOL take-off distance (ft) = 150
- USTOL take-off velocity (mph/ft/sec) = 35/51 .

Class	L/D Climb	L/D Cruise	Disk Loading (lb/ft^2)	Thrust /weight Ratio
Rotor	4.25	4.25	4.5	1.15
VTOL	7.5	10	15	1.15
USTOL	17.5	15		0.34

Follow these steps:

Step 1: Based on the input values find the gross weight of the aircraft.
Step 2: Calculate the battery weight fractions for taxi/take-off, hover, climb, descent, landing and reserve (note that the battery weight fraction for cruise is not included here).
Step 3: Find the battery weight fraction for cruise.
Step 4: Calculate the range.
Step 5: Calculate the needed maximum thrust and power.
Step 6: Estimate the aircraft unit cost.

Details for each step given below.

Calculate Gross Weight (Step 1)

Regardless of aircraft class, the gross weight (Wo) is composed of the weight of the airframe (Wa), total weight of the batteries (Wbt) and the weight of the passengers (Wp). Here, the airframe includes the motors, wiring and controllers, everything but the passengers and batteries.

$$Wo = Wa + Wbt + Wp \quad (1)$$

This can be rewritten as weight fractions:

$$1 = Wa/Wo + Wbt/Wo + Wp/Wo \quad (2)$$

or

$$Wo = Wbt / (1 – Wp/Wo – Wa/Wo) \quad (3)$$

The total weight is the battery weight divided by one minus the passenger weight fraction and the airframe weight fraction.

The Passenger Fraction

$$Wp/Wo = Np * 220/Wo \quad (4)$$

Where Np is the number of passengers including the pilot (if any) all assumed to weigh 220 lb (100 kg).

The Airframe Fraction

Wa/Wo is the airframe fraction. It includes everything but passengers and the batteries. Curve fitting multiple modern composite aircraft with the weight of their engines subtracted out gives an average value of .44. McDonald and German used .55. They were working with 5000 lb vehicles which may have a higher fraction. The X-57 Maxwell will have a gross weight of 3000, has 900 lbs of batteries and, with a crew of 2, 440 lb of payload. This gives an airframe weight fraction of .55. The .44 value is for lighter vehicles and does not include an allowance for the motors and wiring needed. Fitting these two points, then

$$Wa/Wo = .366 * (1 + .0001 * Wo) \quad (5)$$

Battery weight

The total weight of the battery (lb) is a function of the energy density and the battery capacity:

$$Wbt = 2200 * Ec / \rho_b = 2.2 * Np * Ec' \quad (2200 = 2.2\ (lb/kg) * 1000\ (w/kw)) \quad (6)$$

with $Ec' = 1000 * Ec / (Np * \rho_b)$ the battery weight (kg) per passenger.

Gross weight

We now use all the parts to find the gross weight:

$$Wo = Wbt / (1 - Wp/Wo - Wa/Wo) \quad (7)$$

Or

$$Wo = 2.2 * Np * (Ec' + 100) / (1 - Wa/Wo) \quad (8)$$

Can also find the total battery fraction:

$$Wbt/Wo = (1 - Wa/Wo) * (Ec' / (Ec' + 100)) \quad (9)$$

These two equations are critical to the model.

Battery Weight Fractions (Step 2)

With the total weight fraction known, the goal is to find the cruise battery weight or fraction (in bold) based on subtracting the battery weight fraction for all the other segments of the entire mission from the total. In general:

$$Wbt/Wo = Wbto/Wo + Wbh/Wo + Wbc/Wo + Wbcr/Wo + Wbd/Wo + Wblt/Wo + Wbr/Wo \quad (10)$$

So:

$$Wbc/Wo = Wbt/Wo - (Wbto/Wo + Wbh/Wo + Wbcr/Wo + Wbd/Wo + Wblt/Wo + Wbr/Wo) \quad (11)$$

In the next sections the battery weight fractions are estimated in the order they occur.

Battery Weight Fraction for Taxi and Take-Off

For VTOL and rotor-craft it is assumed there is no taxi and take-off power used, so there is no battery weight fraction for them.

For USTOL power is used for taxi and take off includes acceleration, aero drag, and wheel friction. For USTOL with short take-offs, by far the largest of these is the power used to accelerate the mass of the vehicle. This is directly a function of the desired take-off distance and velocity needed. See the discussion in Step 5 for more on this.

For here, the battery fraction used during this phase is very small, < 1% of the total battery so, for all models:

$$Wbto/Wo = 0.0 \quad (12)$$

Battery Weight Fraction for Hover

For a helicopter the power (ft lb/sec) can be found[1] (this will be used again later). First, based on induced velocity through the rotor, the power loss in hover is:

$$P_h = T^{3/2} / (2 * \rho_a * A)^{1/2} \text{ where A is the rotor area.} \quad (13)$$

With thrust = weight (Wo) and the disk loading (DL) = weight divided by the area (Wo/A), this can be rewritten as

$$P_h = Wo * (DL / (2 * \rho_a))^{1/2} \quad (14)$$

Additionally, for rotor craft a figure of merit must included (FOM). This gives the rotor efficiency and is typically about .7, so on a standard day

$$P_h = 14.5 * Wo * DL \,\hat{}.5 * FOM \text{ (ft lb/sec)} \quad (15)$$

[1] W. Johnson, Helicopter Theory, Dover, 1994, pg 31

Energy for hover (ft lb) is the power times the hover time (t_h).

$$E_h = 14.5 * t_h * Wo * DL ^{.5} * FOM \quad 16$$

Or, in watt hours:

$$E_h = .0056 * t_h * Wo * DL ^{.5} * FOM \quad 17$$

With battery energy density ρ_b the battery weight for hover (lbs) is:

$$Wbh = .0056 * t_h * Wo * DL ^{.5} * FOM / \rho_b \quad 18$$

Or for weight fraction

$$Wbh/Wo = .0056 * t_h * DL ^{.5} * FOM / \rho_b \quad 19$$

Battery Weight Fraction for Climb and Descent

Regardless of the type of vehicle, there is the increase in potential and kinetic energy during climb. The potential energy is just the weight (Wo) times the altitude change and the kinetic energy is the one half the mass times the velocity squared. With the efficiency η , then battery energy needed to climb is:

$$E_{cl} = (Wo * AGL + .5 * Wo/ 32.2 * Vc ^ 2) / \eta$$

Or

$$E_{cl} = Wo * (AGL + .0155 * Vc ^ 2) / \eta \quad 20$$

With the assumed values for altitude (AGL =1000 ft), velocity (Vc=150/220/130 (mph/fps/kts))and efficiency (η =.76) ; this reduces to:

$$E_{cl} = Wo * 2302 \text{ (ft lb)} \quad 21$$

With the rate of climb (ROC = 500 ft/min) assumed, the time to climb (tc) can be found and then the power needed is:

$$Pcl1 = .0167 * ROC * Wo * (1 +.0155 * Vc ^ 2 / AGL) / \eta \quad 22$$

Or with the assumed values:

$$Pcl1 = 19.2 * Wo \text{ (ft lb/sec)} \quad 23$$

Power used to overcome aerodynamic drag is (assumed an average velocity):

$$Pcl2 = D * Vavg = Wo * Vavg/ (L/Dc) \quad 24$$

Or with assumed values

$$Pcl2= 110 * Wo * / (L/Dc) \quad \text{(ft lb/sec)} \quad 25$$

So the total power for climb is

$$Pcl = Wo * (110/(L/D) +19.2) \text{ (ft lb/sec)} \qquad 26$$

Energy is P * t where t = 120 sec

$$Ecl = 120 * Wo * ((110/(L/D_{climb}) +19.2) \text{ ftlb} \qquad 27$$

$$Ecl = .045 * Wo * ((110/(L/D_{climb}) +19.2) \text{ (wh)} \qquad 28$$

So battery weight (lbs)

$$Bcl = .099 *Wo * (110/(L/D_{climb}) +19.2)/ \rho_b \qquad 29$$

In terms of weight fraction

$$Bcl/Wo = .099 * (110/(L/D_{climb}) +19.2)/ \rho_b \qquad 30$$

Distance for climb

$$Rto = .5 * t \text{ ^}2 * (220/120) / 5280 = 2.5 \text{ miles} \qquad 31$$

For descent the formula is the same but KE and potential energy are recovered by changing the sign of the second term, so:

$$Bd = .097 *Wo * (110/(L/D) -19.2)/ \rho_b \qquad 32$$

In terms of weight fraction

$$Bcl/Wo = .097 * (110/(L/D) -19.2)/ \rho_b \qquad 33$$

Battery Weight Fraction for Reserve

The additional range for reserve can either be given as a distance or a time. Here, it will be a time (tr) assumed at cruise velocity (Vc) as if traversing to an alternative airport.

So the reserve range (miles) is:

$$Rreserve = tr * Vcruise * 60/5280 \qquad 34$$

Then, using the formula for range as a function of cruise battery fraction, rearrange to get

$$Wbr/Wo = Rreserve/ (.227 * \rho_b * (L/Dcruise) * \eta) \qquad 35$$

$$Wbr/Wo = .050* tr * Vcruise/ (\rho_b * L/Dccruise * \eta) \qquad 36$$

Weight Fraction for Cruise (Step 3)

Since the total battery weight fraction is known from Step 1 and the battery weight fractions for all mission phases except cruise are found in Step 2, the difference is the cruise battery fraction. If positive there is sufficient battery for time at the cruise altitude and velocity. If negative, then the mission cannot be accomplished. With the total battery weight fraction at

$$Wbt/Wo = Wbto/Wo + Wbh/Wo + Wbc/Wo + Wbcr/Wo + Wbd/Wo + Wblt/Wo + Wbr/Wo \quad 37$$

Then:

$$Wbc/Wo = (1 - Wa/Wo) * (Ec' / (Ec' +100)) - (Wbto/Wo + Wbh/Wo + Wbcr/Wo + Wbd/Wo + Wblt/Wo + Wbr/Wo) \quad 38$$

Range (Step 4)

The range is the sum of the climb, descent and cruise ranges. The climb and descent ranges were computed above based on the cruise velocity and time to climb. As mentioned earlier, a 500 ft/min climb rate has been assumed and, with the cruise velocity of 150 mph, the distance covered during climb and descent is 2.5 miles each for all classes of sky taxis.

The cruise range model is built around Breuget's equation for electric airplanes. This was developed by the author for a class in in 2014 and is similar to Hepperle's equation[2].

Power required = Power available. 39

$$\text{Power required} = \text{Drag} * \text{Velocity} = Wo * V / (L/D_{cruise}) \quad 40$$

$$\text{Power available} = Wbcr * \rho_b * \eta / tc \quad 41$$

Based on the battery energy available for cruise.

Equating and realizing that battery energy density is effectively distance (1 wh = 2655 ft lb = 367 kg m):

1 wh/kg = .367 km = 1204 ft 42

Then

$$tc = .367 * Wbcr * \rho_b * \eta * (L/D_{cruise}) / (Wo * V) \quad 43$$

With Range is Vcr * tcr

[2] Hepperle M., Electric Flight – Potential and Limitations, NATO OTAN, STO-MP-AVT-209, 2013

R_{cruise} (km) = .367 * (Wbcr/Wo) * ρ_b * (L/D_{cruise}) * η 44

or

R_{cruise} (miles) = .227 * (Wbcr/Wo) * ρ_b * (L/D_{cruise}) * η 45

So total range is:

R = 2 * R_{climb} + R_{cruise} 46

Thrust and Power Required (Step 5)

Thrust

The needed maximum thrust is a direct function of weight for all classes of vehicle. It is substantially less for USTOL than for rotor-craft and VTOL as explained in the assumptions.

For VTOL and rotors (helicopters)

T/Wo = 1.25 (discussed in the paper) 47

For USTOL it is more complex. The thrust required is the sum of that required for mass acceleration, aerodynamic drag and wheel friction. For these short takeoffs at low speeds the acceleration of the mass is dominant. The acceleration is proportional to the thrust to weight ratio:

T/Wo = a/g where a is the acceleration of the aircraft (assumed constant) and g is the acceleration of gravity.

Or

a = T/Wo * g 49

From basic mechanics, the distance for acceleration (i.e. the ground roll for take-off) is:

Sg = Vto^2 / (2 * a) 50

Or

Sg = Vto^2 / (2 * T/W * g) 51

The velocity for takeoff is when the lift equals the weight, or:

Vto^2 = 2 * Wo / (ρ_a * CL_{to} * A) 52

So

Sg = Wo / (ρ_a * g * CL_{to} * A * T/W) 53

Rearranging then the needed thrust is:

$$T = Wo ^ 2 / (\rho_a * g * CL_{to} * A * Sg) \quad 54$$

Alternatively, based on 49 and 50

$$T/Wo = Vto^2 / (2 * Sg * g) \quad 55$$

Or, with the assumed values for takeoff distance and speed:

$$T/Wo = .27 \quad (USTOL) \quad 56$$

Also, based on 52

$$CL_{to} * A = 2 * Wo / (\rho_a * Vto^2) \quad 57$$

The term (CL_{to} * A) is controlled by the size of the lifting surfaces and the lift generated by them. For PAIs, take-off lift coefficients of 6 and above are possible. Further the lifting surface area can be controlled. Thus the term CL_{to} * A could be treated as an input variable, and the takeoff velocity calculated. For good performance, it should be equal to one third the weight (CLA = Wo/3) of the aircraft. Thus an LSA sized craft where Wo = 1320lbs, a wing area of 100sqft and a takeoff velocity of 51ft/sec then CL at takeoff would need to be 4.3, not unrealistic for a USTOL.

For comparison purposes, the takeoff distance was set at 150 ft. This is seen as adequately short for a pocket airport. Note that it is shorter than the distance for a Carbon Cub, a very high performance LSA STOL.

Power

The maximum power for VTOL and rotor-craft is that used to hover and is a function of the weight, Figure of Merit (FOM) and Disk Loading (DL). For USTOL the maximum power required is that needed for take-off acceleration. The shorter the take-off run, the more power needed.

Power for VTOL and rotor craft is (from the Battery Weight Fraction for Hover)

$$P_h = 14.5 * Wo * DL ^.5 * FOM \text{ (ft lb/sec)} \quad 58$$

For USTOL the power can be found for that needed by the ground run distance (Sg) and the take off velocity (Vto). Specfically:

$$P = force * Vto = mass * acceleration * Vto \quad 59$$

Form basic mechanics, acceleration = Vto^2 / (2 * Sg), so:

$$P = Wo * g * Vto ^2 / (2 * S) \quad 60$$

Aircraft cost (Step 6)

The aircraft cost is composed of the cost of the airframe and the cost of the batteries. The cost of the motors, controllers and wiring are assumed included in the airframe. Since all the classes of aircraft considered will be error to the same degree, this assumption is sufficient. To find the cost only

N (production number) 1000
Labor rate/hr $60/hr
battery cost ($/kwh) Current 200$/kwh, Near Future100$/kwh

Manufacturing hours (Equn 2.4 in Gudmundsson [3])

MfgHrs=9.66 * Wo^0.74 * V_{cruise}^0.54 * N^0.524 * 1.25 61

Manufacturing Hours (Equn 2.90)

MfgCost = =2.1 * MfgHrs * LaborRate 62

Material cost (Equn 2.11)

MatCost = 24.9 *Wo^0.69 * V_{cruise}^0.62 * N^0.79 63

Battery cost

BatCost = B_{kwh} * BC 64

Total cost = MfgCost + MatCost + BatCost 65

[3] Gudmundsson, S., General Aviation Aircraft Design: Applied Methods and Procedures, Butterworth-Heinemann, 2013.